# The combinatorics of orbital varieties closures of nilpotent order 2 in $\mathbf{sl}_n$

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Submitted: Sep 12, 2002; Accepted: Apr 28, 2005; Published: May 6, 2005
Mathematics Subject Classifications: 05E10, 17B10

**Abstract.**

We consider two partial orders on the set of standard Young tableaux. The first one is induced to this set from the weak right order on symmetric group by Robinson-Schensted algorithm. The second one is induced to it from the dominance order on Young diagrams by considering a Young tableau as a chain of Young diagrams. We prove that these two orders of completely different nature coincide on the subset of Young tableaux with 2 columns or with 2 rows. This fact has very interesting geometric implications for orbital varieties of nilpotent order 2 in special linear algebra $sl_n$.

## 1. Introduction

**1.1** Let $\mathbf{S}_n$ be a symmetric group, that is a group of permutations of $\{1, 2, \ldots, n\}$. Respectively, let $\mathbb{S}_n$ be a group of permutations of $n$ positive integers $\{m_1 < m_2 < \ldots < m_n\}$ where $m_i \geq i$. It is obvious that there is a bijection from $\mathbb{S}_n$ onto $\mathbf{S}_n$ obtained by $m_i \to i$, so we will use the notation $\mathbb{S}_n$ in all the cases where the results apply to both $\mathbb{S}_n$ and $\mathbf{S}_n$.

In this paper we write a permutation in a word form

$$w = [a_1, a_2, \ldots, a_n] \ , \quad \text{where } a_i = w(m_i). \qquad (*).$$

All the words considered in this paper are permutations, i.e. with distinct letters only.

Set $p_w(m_i) := j$ if $a_j = m_i$, in other words, $p_w(m_i)$ is the place (index) of $m_i$ in the word form of $w$. (If $w \in \mathbf{S}_n$ then $p_w(i) = w^{-1}(i)$.)

* Supported in part by the Minerva Foundation, Germany, Grant No. 8466

We consider the right weak (Bruhat) order on $\mathbb{S}_n$ that is we put $w \overset{D}{\leq} y$ if for all $i, j \ : \ 1 \leq i < j \leq n$ the condition $p_w(m_j) < p_w(m_i)$ implies $p_y(m_j) < p_y(m_i)$. Note that $[m_1, m_2, \ldots, m_n]$ is the minimal word and $[m_n, m_{n-1}, \ldots, m_1]$ is the maximal word in this order.

**1.2** Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ be a partition of $n$ and $\lambda' := (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_l > 0)$ the conjugate partition, that is $\lambda'_i = \sharp\{j \mid \lambda_j \geq i\}$. In particular, $\lambda'_1 = k$.

We define the corresponding Young diagram $D_\lambda$ of $\lambda$ to be an array of $k$ columns of boxes starting from the top with the $i$-th column containing $\lambda_i$ boxes. Note that it is more customary that $\lambda$ defines the rows of the diagram and $\lambda'$ defines the columns, but in the present context we prefer this convention for the simplicity of notation. Let $\mathbf{D}_n$ denote the set of all Young diagrams with $n$ boxes.

We use the dominance order on partitions. It is a partial order defined as follows. Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\mu = (\mu_1, \cdots, \mu_j)$ be partitions of $n$. Set $\lambda \geq \mu$ if for each $i \ : \ 1 \leq i \leq \min(j, k)$ one has

$$\sum_{m=1}^{i} \lambda_m \geq \sum_{m=1}^{i} \mu_m \ .$$

**1.3** Fill the boxes of the Young diagram $D_\lambda$ with $n$ distinct positive integers $m_1 < m_2 < \ldots < m_n$. If the entries increase in rows from left to right and in columns from top to bottom, we call such an array a Young tableau or simply a tableau. If the numbers in a tableau form the set of integers from 1 to $n$, the tableau is called standard.

Let $\mathbb{T}_n$ denote the set of tableaux with $n$ positive entries $\{m_1 < m_2 < \ldots < m_n\}$ where $m_i \geq i$, and respectively let $\mathbf{T}_n$ denote the set of standard tableaux. Again, the bijection from $\mathbb{T}_n$ onto $\mathbf{T}_n$ is obtained by $m_i \to i$, and we will use the notation $\mathbb{T}_n$ in all the cases where the results apply to both $\mathbb{T}_n$ and $\mathbf{T}_n$. The Robinson-Schensted algorithm (cf. [Sa,§3], or [Kn, 5.1.4], or [F, 4.1] ) gives the bijection $w \mapsto (T(w), Q(w))$ from $\mathbb{S}_n$ onto the set of pairs of tableaux of the same shape. For each $T \in \mathbb{T}_n$ set $\mathcal{C}_T = \{w \mid T(w) = T\}$. It is called a Young cell. The right weak order on $\mathbb{S}_n$ induces a natural order relation $\overset{D}{\leq}$ on $\mathbb{T}_n$ as follows. We say that $T \overset{D}{\leq} S$ if there exists a sequence of tableaux $T = P_1, \ldots, P_k = S$ such that for each $j \ : \ 1 \leq j < k$ there exists a pair $w \in \mathcal{C}_{P_j}, \ y \in \mathcal{C}_{P_{j+1}}$ satisfying $w \overset{D}{\leq} y$.

I would like to explain the notation $\overset{D}{\leq}$ . I use it in honor of M. Duflo who was the first to discover the implication of the weak order on Weyl group for the primitive spectrum of the corresponding enveloping algebra (cf. [D]). I would like to use the notation since his result was the source of my personal interest to the different combinatorial orderings of Young tableaux.

Consider $\mathbf{S}_n$ as a Weyl group of $\mathfrak{sl}_n(\mathbb{C})$. By Duflo, there is a surjection from $\mathbf{S}_n$ onto the set of primitive ideals (with infinitesimal character). Let us define the corresponding primitive ideal by $I_w$. By [D], $w \overset{D}{\leq} y$ implies $I_w \subseteq I_y$. As it was shown by A. Joseph [J], $I_w$ and $I_y$ coincide iff $w$ and $y$ are in the same Young cell. Together these two facts show that the order $\overset{D}{\leq}$ is well defined on $\mathbf{T}_n$.

As shown in [M1, 4.3.1], one may have $T, S \in \mathbf{T}_n$ for which $T \overset{D}{<} S$; yet for any $w \in \mathcal{C}(T),\ y \in \mathcal{C}(S)$ one has $w \overset{D}{\not<} y$. Thus, it is essential to define it through the sequence of tableaux.

**1.4** Take $T \in \mathbb{T}_n$ and let $\mathrm{sh}\,(T)$ be the underlying diagram of $T$. We will write it as $\mathrm{sh}\,(T) = (\lambda_1, \ldots, \lambda_k)$ where $\lambda_i$ is the length of the $i-$th column. Given $i, j\ : 1 \leq i < j \leq n$ we define $\pi_{i,j}(T)$ to be the tableau obtained from $T$ by removing $m_1, \ldots, m_{i-1}$ and $m_{j+1}, \ldots, m_n$ by "jeu de taquin" (cf. [Sch] or 2.10). Put $D_{\langle i,j\rangle}(T) := \mathrm{sh}\,(\pi_{i,j}(T))$. We define the following partial order on $\mathbb{T}_n$ which we call the chain order. We set $T \overset{C}{\leq} S$ if for any $i, j\ :\ 1 \leq i < j \leq n$ one has $D_{\langle i,j\rangle}(T) \leq D_{\langle i,j\rangle}(S)$.

This order is obviously well defined.

**1.5** The above constructions give two purely combinatorial orders on $\mathbb{T}_n$ which are moreover of an entirely different nature.

Given two partial orders $\overset{a}{\leq}$ and $\overset{b}{\leq}$ on the same set S, call $\overset{b}{\leq}$ an extension of $\overset{a}{\leq}$ if $s \overset{a}{\leq} t$ implies $s \overset{b}{\leq} t$ for any $s, t \in S$.

As we explain in 1.11, $\overset{C}{\leq}$ is an extension of $\overset{D}{\leq}$ on $\mathbf{T}_n$. Moreover, these two orders coincide for $n \leq 5$ and $\overset{C}{\leq}$ is a proper extension of $\overset{D}{\leq}$ for $n \geq 6$, as shown in [M].

There is a significant simplification when one considers only tableaux with two columns. Let us denote the subset of tableaux with two columns in $\mathbb{T}_n$ by $\mathbb{T}_n^2$. We show that for $S, T \in \mathbb{T}_n^2$ one has $T \overset{C}{\leq} S$ if and only if $T \overset{D}{\leq} S$. Moreover, for any $T \in \mathbb{T}_n^2$ we construct a canonical representative $w_T \in \mathcal{C}_T$ such that $T \overset{C}{<} S$ if and only if $w_T \overset{D}{<} w_S$.

**1.6** Given a set $S$ and a partial order $\overset{a}{\leq}$, the cover of $t \in S$ in this order is the set of all $s \in S$ such that $t \overset{a}{<} s$ and there is no $p \in S$ such that $t \overset{a}{<} p \overset{a}{<} s$. We will denote it by $\mathcal{D}_a(t)$.

As explained in [M1], in general, even an inductive description of $\mathcal{D}_D(T)$ is a very complex task. Yet, in 3.16 we provide the exact description of $\mathcal{D}_D(T)$ (which is a cover in $\overset{C}{\leq}$ as well) for any $T \in \mathbb{T}_n^2$.

**1.7** For each tableau $T$ let $T^\dagger$ denote the transposed tableau. Obviously, $T \overset{C}{<} S$ iff $S^\dagger \overset{C}{<} T^\dagger$. By Schensted-Schützenberger theorem (cf. 2.14), it is obvious that $T \overset{D}{<} S$ iff $S^\dagger \overset{D}{<} T^\dagger$. Consequently, the above results can be translated to tableaux with two rows.

**1.8** Let us finish the introduction by explaining why these two orders are of interest and what implication our results have for the theory of orbital varieties.

Orbital varieties arose from the works of N. Spaltenstein ([Sp1] and [Sp2]), and R. Steinberg ([St1] and [St2]) during their studies of the unipotent variety of a semisimple group $\mathbf{G}$.

Orbital varieties are the translation of these components from the unipotent variety of $\mathbf{G}$ to the nilpotent cone of $\mathfrak{g} = \mathrm{Lie}\,(\mathbf{G})$. They are defined as follows.

Let $\mathbf{G}$ be a connected semisimple finite dimensional complex algebraic group. Let $\mathfrak{g}$ be its Lie algebra and $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Consider the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$. Fix some triangular decomposition $\mathfrak{g} = \mathfrak{n} \bigoplus \mathfrak{h} \bigoplus \mathfrak{n}^-$. A $\mathbf{G}$ orbit $\mathcal{O}$

in $\mathfrak{g}$ is called nilpotent if it consists of nilpotent elements, that is if $\mathcal{O} = \mathbf{G}\,x$ for some $x \in \mathfrak{n}$. The intersection $\mathcal{O} \cap \mathfrak{n}$ is reducible. Its irreducible components are called orbital varieties associated to $\mathcal{O}$. They are Lagrangian subvarieties of $\mathcal{O}$. According to the orbit method philosophy, they should play an important role in the representation theory of corresponding Lie algebras. Indeed, they play the key role in the study of primitive ideals in $U(\mathfrak{g})$. They also play an important role in Springer's Weyl group representations described in terms of fixed point sets $\mathcal{B}_u$ where $u$ is a unipotent element acting on the flag variety $\mathcal{B}$.

Orbital varieties are very interesting objects from the point of view of algebraic geometry. Given an orbital variety $\mathcal{V}$, one can easily find the nilradical $\mathfrak{m}_{\mathcal{V}}$ of a standard parabolic subalgebra of the smallest dimension containing $\mathcal{V}$. Consider an orbital variety closure as an algebraic variety in the affine linear space $\mathfrak{m}_{\mathcal{V}}$. Then the vast majority of orbital varieties are not complete intersections. So, orbital varieties are examples of algebraic varieties which are both Lagrangian subvarieties and not complete intersections.

**1.9** There are many hard open questions involving orbital varieties. Their only general description was given by R. Steinberg [St1]. Let us explain it briefly.

Let $R \subset \mathfrak{h}^*$ denote the set of non-zero roots, $R^+$ the set of positive roots corresponding to $\mathfrak{n}$ and $\Pi \subset R^+$ the resulting set of simple roots. Let $W$ be the Weyl group for the pair $(\mathfrak{g},\ \mathfrak{h})$. For any $\alpha \in R$ let $X_\alpha$ be the corresponding root space.

For $S, S' \subset R$ and $w \in W$ set $S \cap^w S' := \{\alpha \in S\ :\ \alpha \in w(S')\}$. Then set

$$\mathfrak{n} \cap^w \mathfrak{n} := \bigoplus_{\alpha \in R^+ \cap^w R^+} X_\alpha.$$

This is a subspace of $\mathfrak{n}$. For each closed irreducible subgroup $\mathbf{H}$ of $\mathbf{G}$ let $\mathbf{H}(\mathfrak{n} \cap^w \mathfrak{n})$ be the set of $\mathbf{H}$ conjugates of $\mathfrak{n} \cap^w \mathfrak{n}$. It is an irreducible locally closed subvariety. Let $\overline{*}$ denote the (Zariski) closure of a variety $*$.

Since there are only finitely many nilpotent orbits in $\mathfrak{g}$, it follows that there exists a unique nilpotent orbit which we denote by $\mathcal{O}_w$ such that $\overline{\mathbf{G}(\mathfrak{n} \cap^w \mathfrak{n})} = \overline{\mathcal{O}}_w$.

Let $\mathbf{B}$ be the standard Borel subgroup of $\mathbf{G}$, i.e. such that $\mathrm{Lie}\,(\mathbf{B}) = \mathfrak{b} = \mathfrak{h} \bigoplus \mathfrak{n}$. A result of Steinberg [St1] asserts that $\mathcal{V}_w := \overline{\mathbf{B}(\mathfrak{n} \cap^w \mathfrak{n})} \cap \mathcal{O}_w$ is an orbital variety and that the map $\varphi : w \mapsto \mathcal{V}_w$ is a surjection of $W$ onto the set of orbital varieties. The fibers of this mapping, namely $\varphi^{-1}(\mathcal{V}) = \{w \in W\ :\ \mathcal{V}_w = \mathcal{V}\}$ are called geometric cells.

This description is not very satisfactory from the geometric point of view since a $\mathbf{B}$ invariant subvariety generated by a linear space is a very complex object. For example, one can describe the regular functions (differential operators) on $\overline{\mathcal{V}}_w$ or on $\mathcal{V}_w$ only in some special cases.

**1.10** On the other hand, there exists a very nice combinatorial characterization of orbital varieties in $\mathfrak{sl}_n$ in terms of Young tableaux. Indeed, in that case $\mathcal{V}_w$ and $\mathcal{V}_y$ coincide iff $w$ and $y$ are in the same Young cell. Moreover, let $\mathcal{O}_w = \mathbf{G}\mathcal{V}_w$ be the corresponding nilpotent orbit, then its Jordan form is defined by $\mu = (\mathrm{sh}\,T_w)'$. Let us denote such orbit by $\mathcal{O}_\mu$.

Recall the order relation on Young diagrams from 1.2. A result of Gerstenhaber (see [H, §3.10] for example) describes the closure of a nilpotent orbit.

**Theorem.** *Let $\mu$ be a partition of $n$. One has*

$$\overline{\mathcal{O}}_\mu = \coprod_{\lambda|\lambda\geq\mu} \mathcal{O}_\lambda$$

**1.11** Define geometric order on $\mathbf{T}_n$ by $T \overset{G}{\leq} S$ if $\overline{\mathcal{V}}_S \subset \overline{\mathcal{V}}_T$. In general, the combinatorial description of this order is an open (and very difficult) task. On the other hand, both $\overset{D}{\leq}$ and $\overset{C}{\leq}$ are connected to $\overset{G}{\leq}$ as follows.

Let us identify $\mathfrak{n}$ with the subalgebra of strictly upper-triangular matrices. Any $\alpha \in R^+$ can be decomposed into the sum of simple roots $\alpha = \sum_{k=i}^{j-1} \alpha_k$ where $i < j$. Then the root space $X_\alpha$ is identified with $X_{i,j}$. By [JM, 2.3], $X_{i,j} \in \mathfrak{n} \cap {}^w\,\mathfrak{n}$ if and only if $p_w(i) < p_w(j)$. Thus, $w \overset{D}{\leq} y$ implies $\mathfrak{n} \cap {}^y\,\mathfrak{n} \subset \mathfrak{n} \cap {}^w\,\mathfrak{n}$, hence, also $\overline{\mathcal{V}}_y \subset \overline{\mathcal{V}}_w$ and $\overline{\mathcal{O}}_y \subset \overline{\mathcal{O}}_w$. Therefore, $\overset{G}{\leq}$ is an extension of $\overset{D}{\leq}$ on $\mathbf{T}_n$.

On the other hand, note that $T \overset{G}{\leq} S$ implies, in particular, the inclusion of corresponding orbit closures so that (via Gerstenhaber's construction) $T \overset{G}{\leq} S$ implies $\mathrm{sh}\,(T) \leq \mathrm{sh}\,(S)$. As shown in [M1, 4.1.1], the projections on the Levi factor of standard parabolic subalgebras of $\mathfrak{g}$ preserve orbital variety closures. Moreover, in the case of $\mathfrak{sl}_n$ one has $\pi_{i,j}(\overline{\mathcal{V}}_T) = \overline{\mathcal{V}}_{\pi_{i,j}(T)}$ for any $i, j \,:\, 1 \leq i < j \leq n$ where $\pi_{i,j}(T)$ is obtained from $T$ by jeu de taquin and $\mathcal{V}_{\pi_{i,j}(T)}$ is an orbital variety in the corresponding Levi factor. Thus, $T \overset{G}{\leq} S$ implies $\pi_{i,j}(T) \overset{G}{\leq} \pi_{i,j}(S)$. Altogether, this provides that $\overset{C}{\leq}$ is an extension of $\overset{G}{\leq}$ .

Consequently, $\overset{C}{\leq}$ is an extension of $\overset{G}{\leq}$ and $\overset{G}{\leq}$ is an extension of $\overset{D}{\leq}$ . All three orders coincide for $n \leq 5$, and $\overset{C}{\leq}$ is a proper extension of $\overset{G}{\leq}$ which is, in turn, a proper extension of $\overset{D}{\leq}$ for $n \geq 6$ as shown in [M].

However, our results show that $\overset{D}{\leq}$ and $\overset{C}{\leq}$ coincide on $\mathbf{T}_n^2$ and there they provide a full combinatorial description of $\overset{G}{\leq}$ .

Consider $\mathcal{V}_T$ where $T \in \mathbf{T}_n^2$. For any $X \in \mathcal{V}_T$ one has $X \in \mathcal{O}_{\mathrm{sh}\,(T)}$, that is $X$ is an element of nilpotent order 2 or in other words $X^2 = 0$. Thus, we get a complete combinatorial description of inclusion of orbital varieties closures of nilpotent order 2 in $\mathfrak{sl}_n$.

**1.12** The body of the paper consists of two sections.

In section 2 we explain all the background in combinatorics of Young tableaux essential in the subsequent analysis and set the notation. In particular, we explain Robinson-Schensted insertion from the left and jeu de taquin. I hope this part makes the paper self-contained.

In section 3 we work out the machinery for comparing $\overset{D}{\leq}$ and $\overset{C}{\leq}$ and show that they coincide. The main technical result of the paper is stated in 3.5 and proved in 3.11. Further in 3.12, 3.13 and 3.14 we explain the implications of this result for $\overset{D}{\leq}$, $\overset{G}{\leq}$

and $\overset{C}{\leq}$. In 3.16 we give the exact description of $\mathcal{D}_G(T)$ for $T \in \mathbf{T}_n^2$. Finally, in 3.17 we explain the corresponding facts for the tableaux with two rows.

## 2. Combinatorics of Young tableaux

**2.1** Recall from 1.1 $(*)$ the presentation of $w \in \mathbf{S}_n$ in the word form. Given $w \in \mathbf{S}_n$, set

$$\tau(w) := \{i \ : \ p_w(i+1) < p_w(i)\},$$

that is $\tau(w)$ is the set of left descents of $w$.

Note that if $w \overset{D}{\leq} y$ then $\tau(w) \subseteq \tau(y)$.

**2.2** Given a word or a tableau $*$, we denote by $\langle * \rangle$ the set of its entries. Introduce the following useful notational conventions.

(i) For $m \in \langle w \rangle$ set $w \setminus \{m\}$ to be the word obtained from $w$ by deleting $m$, that is if $m = a_i$ then $w \setminus \{m\} := [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n]$.

(ii) For the words $x = [a_1, \ldots, a_n]$ and $y = [b_1, \ldots, b_m]$ such that $\langle x \rangle \cap \langle y \rangle = \emptyset$ we define a colligation $[x, y] := [a_1, \ldots, a_n, b_1, \ldots, b_m]$.

(iii) For a word $w = [a_1, \ldots, a_n]$ set $\overline{w}$ to be the word with reverse order, that is $\overline{w} := [a_n, a_{n-1}, \ldots, a_1]$.

Given $i, j \ : \ 1 \leq i < j \leq n$, set $\mathbb{S}_{\langle i,j \rangle}$ to be a (symmetric) group of permutations of $\{m_k\}_{k=i}^{j}$. Let us define projection $\pi_{i,j} : \mathbb{S}_n \to \mathbb{S}_{\langle i,j \rangle}$ by omitting all the letters $m_1, \ldots, m_{i-1}$ and $m_{j+1}, \ldots, m_n$ from word $w \in \mathbb{S}_n$, i.e. $\pi_{i,j}(w) = w \setminus \{m_1, \ldots, m_{i-1}, m_{j+1}, \ldots, m_n\}$. For $w \in \mathbf{S}_n$ it is obvious that $\tau(\pi_{i,j}(w)) = \tau(w) \cap \{k\}_{k=i}^{j-1}$.

**Lemma.** *Let $w, y$ be in $\mathbb{S}_n$.*

*(i) For any $a \notin \{m_i\}_{i=1}^n$ one has $w \overset{D}{\leq} y$ iff $[a, w] \overset{D}{\leq} [a, y]$.*

*(ii) For $w, y$ such that $\pi_{1,n-1}(y) = \pi_{1,n-1}(w)$ and $p_w(m_n) = 1,\ p_y(m_n) > 1$ one has $w \overset{D}{>} y$.*

*(iii) $w \overset{D}{<} y$ iff $\overline{y} \overset{D}{<} \overline{w}$.*

*(iv) If $w \overset{D}{\leq} y$ then $\pi_{i,j}(w) \overset{D}{\leq} \pi_{i,j}(y)$ for any $i, j \ : 1 \leq i < j \leq n$.*

All four parts of the lemma are obvious.

**2.3** We will use the following notation for tableaux. Let $T$ be a tableau and let $T_j^i$ for $i, j \in \mathbb{N}$ denote the entry on the intersection of the $i$-th row and the $j$-th column. Given $u$ an entry of $T$, set $r_T(u)$ to be the number of the row, $u$ belongs to and $c_T(u)$ to be the number of the column, $u$ belongs to. Set

$$\tau(T) := \{i \ : \ r_T(i+1) > r_T(i)\}.$$

Let $T_i$ denote the $i$-th column of $T$. Let $\omega_i(T)$ denote the largest entry of $T_i$.

We consider a tableau as a matrix $T := (T_i^j)$ and write $T$ by columns: $T = (T_1, \cdots, T_l)$

For $i, j \ : \ 1 \leq i < j \leq l$ we set $T_{i,j}$ to be a subtableau of $T$ consisting of columns from $i$ to $j$, that is $T_{i,j} = (T_i, \cdots, T_j)$. For each tableau $T$ let $T^\dagger$ denote the transposed tableau. Note that $\mathrm{sh}\,(T^\dagger) = \mathrm{sh}\,(T)'$.

**2.4** Given $D_\lambda \in \mathbf{D}_n$ with $\lambda = (\lambda_1, \cdots, \lambda_j)$, we define a corner box (or simply, a corner) of the Young diagram to be a box with no neighbours to right and below.

For example, in $D$ below all the corner boxes are labeled by $X$.

$$D = \begin{ytableau} \ & \ & \ & X \\ \ & \ & X \\ \ \\ X \end{ytableau}$$

The entry of a tableau in a corner is called a corner entry. Take $D_\lambda$ with $\lambda = (\lambda_1, \cdots, \lambda_k)$. Then there is a corner entry $\omega_i(T)$ at the corner $c$ with coordinates $(\lambda_i, i)$ iff $\lambda_{i+1} < \lambda_i$.

**2.5** We now define the insertion algorithm. Consider a column $C = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$. Given $j \in \mathbb{N}^+ \setminus \langle C \rangle$, let $a_i$ be the smallest entry greater then $j$, if exists. Set

$$j \to C := \begin{cases} \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ j \\ a_{i+1} \\ \vdots \end{pmatrix}, \quad j_C = a_i & \text{if } j < a_r \\ \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ j \end{pmatrix}, \quad j_C = \infty & \text{if } j > a_r \text{ or } C = \emptyset \end{cases}$$

Put also $\infty \to C = C$. The inductive extension of this operation to a tableau $T$ with $l$ columns for $j \in \mathbb{N}^+ \setminus \langle T \rangle$ given by

$$j \Rightarrow T = (j \to T_1, j_{T_1} \Rightarrow T_{2,l})$$

is called the insertion algorithm.

Note that the shape of $j \Rightarrow T$ is the shape of $T$ obtained by adding one new corner. The entry of this corner is denoted by $j_T$.

This procedure (like many others used here) is described in the wonderful book of B.E. Sagan ([Sa]).

**2.6** Let $w = [a_1, a_2, \ldots, a_n]$ be a word. According to Robinson-Schensted algorithm we associate an ordered pair of tableaux $(T(w), Q(w))$ to $w$. The procedure is fully explained

in many places, for example, in [Sa, §3], [Kn, 5.1.4] or [F,4.1]. Here we explain only the inductive procedure of constructing the first tableau $T(w)$ by insertions from the left. In what follows we call it RS procedure.

(1) Set ${}_1T(w) = (a_n)$.
(2) Set ${}_{j+1}T(w) = a_{n+1-j} \Rightarrow {}_jT(w)$ .
(3) Set $T(w) = {}_nT(w)$ .

For example, let $w = [2, 5, 1, 4, 3]$, then

$$ {}_1T(w) = \begin{array}{|c|}\hline 3 \\ \hline\end{array} \qquad {}_2T(w) = \begin{array}{|c|}\hline 3 \\ 4 \\ \hline\end{array} \qquad {}_3T(w) = \begin{array}{ll} 1 & 3 \\ 4 & \end{array} $$

$$ {}_4T(w) = \begin{array}{ll} 1 & 3 \\ 4 & \\ 5 & \end{array} \qquad T(w) = {}_5T(w) = \begin{array}{ll} 1 & 3 \\ 2 & 4 \\ 5 & \end{array} $$

The result due to Robinson and Schensted implies the map $\varphi : w \mapsto T(w)$ is a surjection from $\mathbb{S}_n$ onto $\mathbb{T}_n$.

**2.7** For $T \in \mathbf{T}_n$ one has (cf. for example, [M1, 2.4.14]) $\tau(T(w)) = \tau(w)$. Thus, by 2.1 one has

**Lemma.** *Let $S, T \in \mathbf{T}_n$. If $T \overset{D}{\leq} S$ then $\tau(T) \subseteq \tau(S)$.*

**2.8** Let us describe a few algorithms connected to RS procedure which we use for proofs and constructions.

First let us describe some operations for columns and tableaux. Consider a column $C = \begin{pmatrix} a_1 \\ \vdots \end{pmatrix}$.

(i) For $m \in \langle C\rangle$ set $C \setminus \{m\}$ to be a column obtained from $C$ by deleting $m$.
(ii) For $j \in \mathbb{N},\ j \notin \langle C\rangle$ set $C + \{j\}$ to be a column obtained from $C$ by adding $j$ at the right place of $C$, that is if $a_i$ is the greatest element of $\langle C\rangle$ smaller than $j$ then $C + \{j\}$ is obtained from $C$ by adding $j$ between $a_i$ and $a_{i+1}$.
(iii) We define a pushing left operation. Again let $j \in \mathbb{N},\ j \notin \langle C\rangle$ and $j > a_1$. Let $a_i$ be the greatest entry of $C$ smaller than $j$ and set :

$$ C \leftarrow j := \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ j \\ a_{i+1} \\ \vdots \end{pmatrix}, \qquad j^C := a_i. $$

The last operation is extended to a tableau $T$ by induction on the number of columns. Let $T_m$ be the last column of $T$ and assume $T_m^1 < j$. Then $T \leftarrow j = (T_{1,m-1} \leftarrow j^{T_m}, T_m \leftarrow j)$. We denote by $j^T$ the element pushed out from the first column of the tableau in the last step.

**2.9** The pushing left operation gives us a procedure of deleting a corner inverse to the insertion algorithm. This is also described in many places, in particular, in all three books mentioned above.

As a result of insertion, we get a new tableau of a shape obtained from the old one just by adding one corner. As a result of deletion, we get a new tableau of a shape obtained from the old one by removing one corner.

Let $T = (T_1, \ldots, T_l)$. Recall the definition of $\omega_i(T)$ from 2.3. Assume $\lambda_i > \lambda_{i+1}$ and let $c = c(\lambda_i, i)$ be a corner of $T$ on the $i$-th column. To delete the corner $c$ we delete $\omega_i(T)$ from the column $T_i$ and push it left through the tableau $T_{1,i-1}$. The element pushed out from the tableau is denoted by $c^T$. This is written

$$T \Leftarrow c := (T_{1,i-1} \leftarrow \omega_i(T), T_i \setminus \{\omega_i(T)\}, T_{i+1,l})$$

For example,

$$\begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 5 & \end{array} \Leftarrow c(2,2) = \begin{array}{cc} 1 & 3 \\ 4 & \\ 5 & \end{array}, \qquad c^T = 2.$$

Note that insertion and deletion are indeed inverse since for any $T \in \mathbb{T}_n$

$$c^T \Rightarrow (T \Leftarrow c) = T \quad \text{and} \quad (j \Rightarrow T) \Leftarrow j_T = T \quad (\text{for } j \notin \langle T \rangle)$$

Note that sometimes we will write $T \Leftarrow a$ where $a$ is a corner entry just as we have written above.

Let $\{c_i\}_{i=1}^{j}$ be a set of corners of $T$. By Robinson-Schensted procedure, one has

$$\mathcal{C}_T = \coprod_{i=1}^{j} \coprod_{y' \in \mathcal{C}_{T \Leftarrow c_i}} [c_i^T, y']. \qquad (*)$$

**2.10** Let us describe the jeu de taquin procedure (see [Sch]) which removes $T_j^i$ from $T$. The resulting tableau is denoted by $T \setminus \{T_j^i\}$. The idea of jeu de taquin is to remove $T_j^i$ from the tableau and to fill the gape created so that the resulting object is again a tableau. The procedure goes as following. Remove a box from the tableau. Examine the content of the box to the right of the removed box and that of the box below of the removed box. Slide the box containing the smaller of these two numbers to the vacant position. Now repeat this procedure to fill the hole created by the slide. Repeat the process until no holes remain, that is until the hole has worked itself to the corner of the tableau.

The result due to M. P. Schützenberger [Sch] gives

**Theorem.** *If $T$ is a Young tableau then $T\setminus\{T^i_j\}$ is a Young tableau and the elimination of different entries from $T$ by jeu de taquin is independent of the order chosen.*

Therefore, given $i_1,\ldots,i_s\in\langle T\rangle$, a tableau $T\setminus\{i_1,\ldots,i_s\}$ is a well defined tableau. For example, let us take

$$T=\begin{array}{ccc} 1 & 2 & 5\\ 3 & 4 & \\ 6 & & \end{array}$$

Then a few tableaux obtained from $T$ by jeu de taquin are

$$T\setminus\{6\}=\begin{array}{ccc} 1 & 2 & 5\\ 3 & 4 & \end{array}\ ,\quad T\setminus\{3\}=\begin{array}{ccc} 1 & 2 & 5\\ 4 & & \\ 6 & & \end{array}\ ,\quad T\setminus\{1,2\}=\begin{array}{ccc} 3 & 4 & 5\\ 6 & & \end{array}\ .$$

**2.11** Given $s,t\ :\ 1\le s<t\le n$, set $\mathbb{T}_{\langle s,t\rangle}$ to be a set of Young tableaux with the entries $\{m_k\}_{k=s}^t$. Let us define projection $\pi_{s,t}:\mathbb{T}_n\to\mathbb{T}_{\langle s,t\rangle}$ by $\pi_{s,t}(T)=T\setminus\{m_1,\ldots,m_{s-1},m_{t+1},\ldots,m_n\}$. As a straightforward corollary of 2.10 (cf. for example, [M1, 4.1.1]), we get

**Theorem.** *for any $s,t\ :\ 1\le s<t\le n$ one has $\pi_{s,t}(T(w))=T(\pi_{s,t}(w))$.*

**2.12** As a straightforward corollary of lemma 2.2 (iv) and theorem 2.11, we get that $\stackrel{D}{\le}$ is preserved under projections and, as a straightforward corollary of lemma 2.2 (i) and RS procedure, we get that $\stackrel{D}{\le}$ is preserved under insertions, namely

**Proposition.** *Let $T,S$ be in $\mathbb{T}_n$. If $T\stackrel{D}{\le}S$ then*

*(i) for any $s,t\ :\ 1\le s<t\le n$ one has $\pi_{s,t}(T)\stackrel{D}{\le}\pi_{s,t}(S)$.*

*(ii) for any $a\notin\{m_s\}_{s=1}^n$ one has $a\Rightarrow T\stackrel{D}{\le}a\Rightarrow S$.*

**2.13** Consider $T\in\mathbf{T}_n$. Note that

$$\pi_{i,i+1}(T)=\begin{cases}\begin{array}{|c|}\hline i\\ i{+}1\\ \hline\end{array} & \text{if } i\in\tau(T)\\[2em] \begin{array}{|cc|}\hline i & i{+}1\\ \hline\end{array} & \text{if } i\notin\tau(T)\end{cases}$$

We need the following properties of the chain order.

**Proposition.** *Let $S, T \in \mathbf{T}_n$.*

*(i) If $T \overset{C}{\leq} S$ then $\tau(T) \subset \tau(S)$.*

*(ii) If $T \overset{C}{\leq} S$ then for any $i, j \ : \ 1 \leq i < j \leq n$ one has $\pi_{i,j}(T) \overset{C}{\leq} \pi_{i,j}(S)$.*

*(iii) If $T \overset{D}{\leq} S$ then $T \overset{C}{\leq} S$.*

**Proof.**
The first two assertions are trivial. The third assertion is a corollary of Steinberg's construction explained in 1.9 and 1.10 and of proposition 2.12 (i) or of the results explained in 1.11.

Indeed, $y \overset{D}{\leq} w$ implies that $\overline{\mathcal{O}}_y \supseteq \overline{\mathcal{O}}_w$. Thus, $T \overset{D}{\leq} S$ implies $\mathrm{sh}\,(T) \leq \mathrm{sh}\,(S)$. By proposition 2.12 (i), $T \overset{D}{\leq} S$ implies $\pi_{s,t}(T) \overset{D}{\leq} \pi_{s,t}(S)$ for any $s, t \ : \ 1 \leq s < t \leq n$. Altogether, this provides $T \overset{C}{\leq} S$.

∎

**2.14** All the results for the tableaux with two columns can be translated to tableaux with two rows by Schensted-Schützenberger theorem (cf. [Kn, 5.4.1]).

**Theorem.** *For any $w \in \mathbb{S}_n$ one has $T^\dagger(w) = T(\overline{w})$.*

## 3. Combinatorics of $\mathbf{T}_n^2$.

**3.1** Recall from 1.5 that $\mathbb{T}_n^2 \subset \mathbb{T}_n$ is the set of Young tableaux with 2 columns. For $T \in \mathbb{T}_n^2$ let $\lambda_1(T)$ be the length of the first column and $\lambda_2(T)$ be the length of the second column, that is $\mathrm{sh}\,(T) = (\lambda_1(T), \lambda_2(T))$.

**Lemma.** *Let $T \in \mathbf{T}_n^2$ be such that $c_T(n) = 2$. Set $T' = \pi_{2,n}(T)$. Then $c_{T'}(n) = 2$ if and only if either $\lambda_1(T) > \lambda_2(T)$ or there exists $i$ such that $T_2^i < T_1^{i+1}$.*

The proof is a straightforward and easy computation, so we omit it.

**3.2** Consider tableaux $T, S \in \mathbf{T}_n^2$.

**Lemma.** *If $c_T(n) = 1$ and $S \overset{C}{>} T$ then $c_S(n) = 1$.*

**Proof.**
This is true for $n = 3$. Assume this is true for $k = n-1$ and show for $k = n$. If $c_T(n) = 1$ then $\lambda_1(T) > \lambda_2(T)$. Since $S \overset{C}{>} T$ one has $\mathrm{sh}\,(S) > \mathrm{sh}\,(T)$. Thus, $\lambda_1(S) > \lambda_2(S)$. Assume $c_S(n) = 2$. Then by lemma 3.1 $c_{\pi_{2,n}(S)}(n) = 2$. On the other hand, $c_{\pi_{2,n}(T)}(n) = 1$ by the induction assumption, and this is a contradiction.

∎

**3.3** As a corollary of lemma 3.2, we get

**Corollary.** *For $S, T \in \mathbf{T}_n^2$ one has*

*(i) If $T \neq S$ and $\mathrm{sh}\, T = \mathrm{sh}\, S$ then $T$ and $S$ are incompatible in the chain order.*

*(ii) If $S \overset{C}{>} T$ then $\langle S_1 \rangle \supset \langle T_1 \rangle$ and $\langle S_2 \rangle \subset \langle T_2 \rangle$.*

**Proof.**

(i) This is true for $n = 3$. Assume this is true for $n-1$ and show for $n$.

(a) If $c_T(n) = c_S(n)$ then $\pi_{1,n-1}(T) \neq \pi_{1,n-1}(S)$ and $\operatorname{sh}\pi_{1,n-1}(T) = \operatorname{sh}\pi_{1,n-1}(S)$, hence, they are incompatible by assumption hypothesis.

(b) If $c_T(n) = 1$ and $c_S(n) = 2$ then $\operatorname{sh}\pi_{1,n-1}(T) = (\lambda_1(T) - 1, \lambda_2(T))$ and $\operatorname{sh}\pi_{1,n-1}(S) = (\lambda_1(T), \lambda_2(T) - 1)$ so that $\operatorname{sh}\pi_{1,n-1}(T) < \operatorname{sh}\pi_{1,n-1}(S)$. Hence, $S \overset{C}{\not\leq} T$. On the other hand, by lemma 3.2 $T \overset{C}{\not\leq} S$.

(ii) For any $j \ : \ j < n$ one has $\pi_{1,j}(S) \overset{C}{\geq} \pi_{1,j}(T)$. If $c_T(j) = 1$ then by lemma 3.2 applied to $\pi_{1,j}(T),\ \pi_{1,j}(S)$ we get $c_S(j) = 1$. Further note that $\langle T_2 \rangle = \{i\}_{i=1}^n \setminus \langle T_1 \rangle$.

∎

Note that in general neither of these assertion is true, as it is shown in the following example: $T \overset{C}{<} S$ where

$$T = \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \end{array} \qquad \text{and} \qquad S = \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & & \end{array}$$

Therefore, to avoid two tableaux of the same shape to be in the chain order we have to restrict the chain order by the demand that if for some $T \overset{C}{<} S$ and for some $i, j \ : \ 1 \leq i < j \leq n$ one has $D_{\langle i,j \rangle}(T) = D_{\langle i,j \rangle}(S)$ then $\pi_{i,j}(T) = \pi_{i,j}(S)$. As we see, we do not need this restriction on $\mathbf{T}_n^2$.

**3.4** One has

**Lemma.** *If $T \overset{D}{<} S$ and $c_T(n) = 2,\ c_S(n) = 1$ then $T' \overset{D}{\leq} S$ where $T' = (T_1 + \{n\}, T_2 \setminus \{n\})$.*

**Proof.**

Indeed, if $T \overset{D}{<} S$ then by proposition 2.12 (i) $\pi_{1,n-1}(T) \overset{D}{\leq} \pi_{1,n-1}(S)$ and further by proposition 2.12 (ii) $(T_1 + \{n\}, T_2 \setminus \{n\}) = n \Rightarrow \pi_{1,n-1}(T) \overset{D}{\leq} n \Rightarrow \pi_{1,n-1}(S) = S$.

∎

**3.5** Now we construct the special representative of $\mathcal{C}_T$ which plays the key role in our constructions.

Given $T \in \mathbf{T}_n^2$, put $T_{(n)} = T$. Let $z_i := \max\langle T_{(i)} \rangle$. Obviously $z_i$ is a corner element of $T_{(i)}$. Set $T_{(i-1)} = T_{(i)} \Leftarrow z_i$. Recall notion $c^T$ from 2.9. Set $a_i := z_i^T$.

Note that for any $s \in T_2$ there exists a unique $i$ such that $s = z_i = z_{i-1}$. For $s \in T_2$ set $T\{s\} := T_{(i)}$.

For example, let

$$T = \begin{array}{|cc|}\hline 1 & 3 \\ 2 & 5 \\ 4 & 6 \\ \cline{2-2} 7 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

then

$$T_{(7)} = T,\ z_7 = 7; \quad T_{(6)} = T_{(7)} \Leftarrow 7 = \begin{array}{|cc|}\hline 1 & 3 \\ 2 & 5 \\ 4 & 6 \\ \hline \end{array},\ a_7 = 7,\ z_6 = 6,\ T\{6\} = T_{(6)};$$

$$T_{(5)} = T_{(6)} \Leftarrow 6 = \begin{array}{|cc|}\hline 1 & 3 \\ 2 & 5 \\ \cline{2-2} 6 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array},\ a_6 = 4,\ z_5 = 6;$$

$$T_{(4)} = T_{(5)} \Leftarrow 6 = \begin{array}{|cc|}\hline 1 & 3 \\ 2 & 5 \\ \hline \end{array},\ a_5 = 6,\ z_4 = 5,\ T\{5\} = T_{(4)};$$

$$T_{(3)} = T_{(4)} \Leftarrow 5 = \begin{array}{|cc|}\hline 1 & 3 \\ \cline{2-2} 5 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array},\ a_4 = 2,\ z_3 = 5;$$

$$T_{(2)} = T_{(3)} \Leftarrow 5 = \begin{array}{|cc|}\hline 1 & 3 \\ \hline \end{array},\ a_3 = 5,\ z_2 = 3,\ T\{3\} = T_{(2)};$$

$$T_{(1)} = T_{(2)} \Leftarrow 3 = \begin{array}{|c|}\hline 3 \\ \hline \end{array},\ a_2 = 1,\ z_1 = a_1 = 3.$$

Put $w_T := [a_n, a_{n-1}, \ldots, a_1]$. In our example $w_T = [7, 4, 6, 2, 5, 1, 3]$. Note that by Robinson-Schensted procedure $T(w_T) = T$.

Now we can formulate the main theorem of the paper

**Theorem.** *For $T, S \in \mathbf{T}_n^2$ one has $T \overset{C}{<} S$ iff $w_T \overset{D}{<} w_S$.*

To prove the theorem we need a few technical lemmas.

**3.6** First of all we show that $w_T$ is a maximal element of $\mathcal{C}_T$ in the weak order.

**Lemma.** *For any $y \in \mathcal{C}_T$ one has $y \overset{D}{\leq} w_T$.*

**Proof.**
This is true for $n = 3$. Assume that this is true for $k \leq n-1$ and show for $n$. Take $T \in \mathbf{T}_n^2$. Set $\omega_1 := \omega_1(T)$ and $\omega_2 := \omega_2(T)$.

(i) If $c_T(n) = 1$ (which means $\omega_1 = n$) then $w_T = [n, w_{\pi_{1,n-1}(T)}]$ and for any $y$ such that $T(\pi_{1,n-1}(y)) = \pi_{1,n-1}(T)$ one has by lemma 2.2 (ii) $y \overset{D}{\leq} [n, \pi_{1,n-1}(y)]$. In particular, for any $y \in \mathcal{C}_T$ one has $y \overset{D}{\leq} [n, \pi_{1,n-1}(y)] \overset{D}{\leq} [n, w_{\pi_{1,n-1}(T)}]$ just by induction assumption and lemma 2.2 (i).

(ii) If $c_T(n) = 2$ (which means $\omega_2 = n$) then $\omega_1^T = \omega_2^T = \omega_1$ thus, by 2.9 $(*)$ any $y \in \mathcal{C}_T$ has a form $y = [\omega_1, y']$ where either $T(y') = T \Leftarrow n =: T'$ or $T(y') = T \Leftarrow \omega_1 =: T''$. Note that $w_T = [\omega_1, w_{T'}]$ thus, by induction assumption and lemma 2.2 (i) for any $y' \in \mathcal{C}_{T'}$ one has $[\omega_1, y'] \overset{D}{\leq} w_T$. For any $y' \in \mathcal{C}_{T''}$ one has just by induction assumption that $y' \overset{D}{\leq} w_{T''}$ where $w_{T''} = [\omega_1(T''), n, z]$, where $z$ is the rest of this word. Note that by definition of the right weak order $w_{T''} \overset{D}{<} [n, \omega_1(T''), z]$ so that for any $y' \in \mathcal{C}(T'')$ one has $y' \overset{D}{<} [n, \omega_1(T''), z]$. On the other hand, $T([n, \omega_1(T''), z]) = T'$. Indeed, $T(\pi_{1,n-2}([n, \omega_1(T''), z])) = \pi_{1,n-2}(T'') = \pi_{1,n-2}(T')$ and by RS procedure $c_{T([n,\omega_1(T''),z])}(n) = 1$. Thus, for any $y' \in \mathcal{C}(T'')$ one has $y' \overset{D}{<} [n, \omega_1(T''), z] \overset{D}{\leq} w_{T'}$. Applying lemma 2.2 (i) we get that for any $y' \in \mathcal{C}(T'')$ one has $[\omega_1, y'] \overset{D}{<} w_T$.

■

**3.7** As a corollary of lemma 3.6 and its proof, we get

**Corollary.** *If $c_T(n) = 2$ and $T' = n \Rightarrow \pi_{1,n-1}(T)$ then for any $y \in \mathcal{C}_T$ one has that $y \overset{D}{<} w_{T'}$.*

**Proof.**
Indeed, $y \overset{D}{\leq} w_T = [\omega_1(T), n, z] \overset{D}{<} [n, \omega_1(T), z]$ and as we have shown in (ii) of the proof of lemma 3.6, $[n, \omega_1(T), z] \in \mathcal{C}(T')$, hence, by lemma 3.6, $y \overset{D}{<} w_{T'}$.

■

**3.8** Let us return to the description of the orders on the level of tableaux.

**Lemma.** *Let $T, S \in \mathbf{T}_n^2$. If $S \overset{C}{>} T$ and $c_T(n) = c_S(n)$ then $\omega_1(T) = \omega_1(S)$.*

**Proof.**
If $c_T(n) = 1$ then $\omega_1(T) = \omega_1(S) = n$.

Assume that $c_T(n) = 2$. For $n = 4$ this is true. Assume this is true for $k = n-1$ and show for $k = n$. Consider $T' = \pi_{1,n-1}(T)$ and $S' = \pi_{1,n-1}(S)$. By proposition 2.13 (ii), $S' \overset{C}{>} T'$.

(i) If $c_{S'}(n-1) = 2$ then by lemma 3.2 $c_{T'}(n-1) = 2$ and by induction assumption $\omega_1(S') = \omega_1(T')$. On the other hand, $\omega_1(S') = \omega_1(S)$ and $\omega_1(T') = \omega_1(T)$.

(ii) If $c_{S'}(n-1) = 1$ then $n-1 \notin \tau(S)$. Thus, by proposition 2.13 (i) $n-1 \notin \tau(T)$ so that $c_{T'}(n-1) = 1$ and $\omega_1(T) = \omega_1(S) = n-1$.

■

**3.9** Let $S$ be a tableau with two columns. For $x \in \langle S_2 \rangle$ recall notion $S\{x\}$ from 3.5. Since $x$ is $\omega_2(S\{x\})$ we consider $S\{x\} \Leftarrow x$ and get $x^{S\{x\}}$ (as defined in 2.9). Obviously, $x^{S\{x\}}$ is some element of $S_1$.

**Lemma.** *Let* $T, S \in \mathbf{T}_n^2$. $w_T \overset{D}{<} w_S$ *iff* $\langle S_2 \rangle \subset \langle T_2 \rangle$ *and for any* $x \in \langle S_2 \rangle$ *one has* $x^{S\{x\}} \in \langle T_1 \rangle$.

**Proof.**
First of all note that $w_T \overset{D}{<} w_S$ implies that $\langle S_2 \rangle \subset \langle T_2 \rangle$ by corollary 3.3 (ii). As well, this implies that for any $x \in \langle S_2 \rangle$ one has $x^{S\{x\}} \in \langle T_1 \rangle$. Indeed, assume that there exist $x \in \langle S_2 \rangle$ such that $a := x^{S\{x\}} \notin \langle T_1 \rangle$. Then $c_T(a) = 2$ and by definition of $w_T$ one has $p_{w_T}(a) > p_{w_T}(x)$. On the other hand, $p_{w_S}(a) = p_{w_S}(x) - 1$. Thus, we have found $a < x$ such that $p_{w_T}(x) < p_{w_T}(a)$ and $p_{w_S}(x) > p_{w_S}(a)$. This implies that $w_T \overset{D}{\not\leq} w_S$.

We show the other direction by induction. The claim is true for $n = 4$. Assume this is true for $k \leq n-1$ and show for $k = n$.

(i) If $c_T(n) = 1$ then $c_S(n) = 1$ since $\langle S_2 \rangle \subset \langle T_2 \rangle$. Set $T' := \pi_{1,n-1}(T)$ and $S' := \pi_{1,n-1}(S)$. One has $T_1' = T_1 - n$ and $S_2' = S_2$. As well, $x^{S\{x\}} = x^{S'\{x\}}$ for any $x \in \langle S_2 \rangle$. Thus, if $x^{S\{x\}} \in \langle T_1 \rangle$ then $x^{S'\{x\}} \in \langle T_1' \rangle$. By induction hypothesis, this provides $w_{T'} \overset{D}{<} w_{S'}$. Note that $w_T = [n, w_{T'}],\ w_S = [n, w_{S'}]$. Thus, by lemma 2.2 (i) $w_T \overset{D}{<} w_S$.

(ii) Assume that $c_T(n) = c_S(n) = 2$. Since $\langle S_2 \rangle \subset \langle T_2 \rangle$, we get that $\langle S_1 \rangle \supset \langle T_1 \rangle$ and, in particular, $\omega_1(T) \leq \omega_1(S)$. Since $n^S = \omega_1(S)$, by the condition $n^S \in \langle T_1 \rangle$ we get that $\omega_1(T) = \omega_1(S)$. Let us denote it by $\omega_1$. Thus, by the construction $w_T = [\omega_1, n, w_{T'}]$ and $w_S = [\omega_1, n, w_{S'}]$ where $T' = (T_1 - \omega_1, T_2 - n)$ and $S' = (S_1 - \omega_1, S_2 - n)$. Let us show that $T', S'$ satisfy the conditions. It is obvious that $\langle S_2' \rangle \subset \langle T_2' \rangle$. Further, $n^S = \omega_1 \in \langle T_1 \rangle$ and for any $x\ : x \neq n,\ x \in \langle S_2 \rangle$ one has $x \in \langle S_2' \rangle$ and $x^{S\{x\}} = x^{S'\{x\}}$ just by construction. Moreover, for such $x$ one has $x^{S\{x\}} \neq \omega_1$. Thus, the condition $x^{S\{x\}} \in \langle T_1 \rangle$ for any $x \in \langle S_2 \rangle$ provides $x^{S'\{x\}} \in \langle T_1' \rangle$ for any $x \in \langle S_2' \rangle$. By induction hypothesis, this implies $w_{T'} \overset{D}{<} w_{S'}$. Again by lemma 2.2 (i) if $w_{T'} \overset{D}{<} w_{S'}$ then $w_T \overset{D}{<} w_S$.

(iii) Finally, assume that $c_T(n) = 2$ and $c_S(n) = 1$. Consider $T' = n \Rightarrow \pi_{1,n-1}(T)$. Note that $\langle S_2 \rangle \subset \langle T_2 \rangle$ and $n \notin \langle S_2 \rangle$ imply that $\langle S_2 \rangle \subset \langle T_2' \rangle$. Let us show that $T', S$ satisfy the second condition as well. Indeed, $T_1' = (T_1 + n)$. Thus, for any $x \in \langle S_2 \rangle$ one has $x^{S\{x\}} \in \langle T_1 \rangle$ iff $x^{S\{x\}} \in \langle T_1' \rangle$. By (i), this implies $w_S \overset{D}{>} w_{T'}$ and by corollary 3.7 $w_{T'} \overset{D}{>} w_T$. This completes the proof.

∎

**3.10** We need the following result about the chain order

**Lemma.** *Let* $T, S \in \mathbf{T}_n$. *Let* $T \overset{C}{<} S$ *and assume that* $c_T(n) = c_S(n) = 2$. *If* $T' = T \Leftarrow n,\ S' = S \Leftarrow n$ *then* $T' \overset{C}{<} S'$.

**Proof.**
By lemma 3.8, the assumption $c_T(n) = c_S(n) = 2$ implies $\omega_1(T) = \omega_1(S)$ and we will denote it by $\omega_1$. We give a proof by induction. This is true for $n = 4$. Assume this is true for $k = n - 1$ and show for $k = n$.

(i) Suppose that $\omega_1 = n - 1$ then $T'$ is equivalent to $\pi_{1,n-1}(T)$ and $S'$ is equivalent to $\pi_{1,n-1}(S)$ and the statement is obvious.

(ii) Let us consider the case $\omega_1 < n-1$. We have that $T', S' \in \mathbb{T}^2_{n-1}$. To show that $T' \overset{C}{<} S'$ we note first that $\mathrm{sh}\,(T') = (\lambda_1(T), \lambda_2(T) - 1)$ and $\mathrm{sh}\,(S') = (\lambda_1(S), \lambda_2(S) - 1)$. Thus, $\mathrm{sh}\,(T') < \mathrm{sh}\,(S')$. As well, $\mathrm{sh}\,(\pi_{1,n-2}(T')) = (\lambda_1(T) - 1, \lambda_2(T) - 1)$ and $\mathrm{sh}\,(\pi_{1,n-2}(S')) = (\lambda_1(S)-1, \lambda_2(S)-1)$. Thus, again, $\mathrm{sh}\,(\pi_{1,n-2}(T')) < \mathrm{sh}\,(\pi_{1,n-2}(S'))$. Let us show that $\pi_{1,n-3}(T') \overset{C}{<} \pi_{1,n-3}(S')$ by induction hypothesis. Indeed, $P = T, S$ and for $P' = T', S'$ one has $\pi_{1,n-3}(P') = \pi_{1,n-3}(\pi_{1,n-1}(P) \Leftarrow n - 1)$ so that by induction hypothesis $\pi_{1,n-3}(T') \overset{C}{<} \pi_{1,n-3}(S')$. To complete the proof we have to show that $\pi_{2,n-1}(T') \overset{C}{\leq} \pi_{2,n-1}(S')$. Indeed, set $P'' = \pi_{2,n}(P)$ where $P$ is $T$ or $S$. Since $S \overset{C}{>} T$ one has $\lambda_1(S) > \lambda_2(S)$. Thus, $S$ satisfies conditions (i) and (ii) of lemma 3.1, so that $c_{S''}(n) = 2$. This implies in turn by lemma 3.2 that $c_{T''}(n) = 2$. In particular, this provides $\pi_{2,n-1}(P') = P'' \Leftarrow n$. Hence, $\pi_{2,n-1}(T') \overset{C}{\leq} \pi_{2,n-1}(S')$ by induction assumption.

■

Note that this property is unique for $\mathbf{T}^2_n$. Indeed, in general, the facts $T \overset{C}{<} S$ and $c_T(n) = c_S(n)$ even do not provide that $\langle T \Leftarrow n\rangle = \langle S \Leftarrow n\rangle$.

**3.11** Now we are ready to prove theorem 3.5. Let us recall its formulation.

**Theorem.** *For $T, S \in \mathbf{T}^2_n$ one has $T \overset{C}{<} S$ iff $w_T \overset{D}{<} w_S$.*

**Proof.**
As we explained in 1.11, $w_T \overset{D}{<} w_S$ implies $T \overset{C}{<} S$.

We will show the other direction by induction. For $n = 3$ the other direction is true. Assume that for $k \leq n - 1$ if $T \overset{C}{<} S$ then $w_T \overset{D}{<} w_S$ and show this for $k = n$.

Assume $T \overset{C}{<} S$.

(i) If $c_S(n) = 1$ then consider $S' = \pi_{1,n-1}(S)$ and $T' = \pi_{1,n-1}(T)$. By proposition 2.13 (ii), $T' \overset{C}{\leq} S'$, thus, by induction assumption $w_{T'} \overset{D}{\leq} w_{S'}$. One has $w_S = [n, w_{S'}]$ and by lemma 2.2 (i) this implies $[n, w_{S'}] \overset{D}{\geq} [n, w_{T'}] = w_{T''}$ where $T'' = n \Rightarrow T'$. By corollary 3.7, $w_{T''} \overset{D}{\geq} w_T$. Thus, we get in that case $w_T \overset{D}{\leq} w_S$.

(ii) If $c_S(n) = 2$ then by lemma 3.2 $c_T(n) = 2$. By lemma 3.8, $\omega_1(T) = \omega_1(S) =: \omega_1$. As well, $\omega_1 = n^S = n^T$. Consider $S' = S \Leftarrow n$ and $T' = T \Leftarrow n$. By lemma 3.10, $T' \overset{C}{<} S'$ and by induction hypothesis this provides $w_{T'} \overset{D}{<} w_{S'}$. On the other hand, $w_T = [\omega_1, w_{T'}]$ and $w_S = [\omega_1, w_{S'}]$. Thus, by lemma 2.2 (i) in that case, as well, $w_T \overset{D}{<} w_S$.

■

**3.12** The first and very easy corollary of the theorem is

**Corollary.** *For $T, S \in \mathbf{T}_n^2$ one has $T \overset{D}{<} S$ iff $w_T \overset{D}{<} w_S$.*

**Proof.**
The implication $w_T \overset{D}{<} w_S \ \Rightarrow\ T \overset{D}{<} S$ follow just from the definition; the other implication is obvious from theorem 3.11 since $T \overset{D}{<} S$ implies $T \overset{C}{<} S$.

∎

**3.13** As well, we get the following geometric fact from this purely combinatorial theorem

**Corollary.** *For orbital varieties $\mathcal{V}_T, \mathcal{V}_S$ of nilpotent order 2 one has $\mathcal{V}_S \subset \overline{\mathcal{V}}_T$ if and only if $\mathfrak{n} \cap^{w_S} \mathfrak{n} \subset \mathfrak{n} \cap^{w_T} \mathfrak{n}$ that the inclusion of orbital variety closures is determined by inclusion of generating subspaces.*

**Proof.**
Again one implication is obvious from the definition and the other one from theorem 3.11 since $\mathcal{V}_S \subset \overline{\mathcal{V}}_T$ implies by 1.11 $S \overset{C}{>} T$.

∎

**3.14** Theorem 3.11 and corollary 3.12 provide us also

**Corollary.** *$\overset{C}{\leq}$ and $\overset{D}{\leq}$ coincide on orbital varieties of nilpotent order 2.*

**3.15** Note that lemma 3.9 together with theorem 3.11 give the exact description of inclusion of orbital variety closures of nilpotent order 2 in terms of Young tableaux. Since $\overset{G}{\leq}$, $\overset{D}{\leq}$, and $\overset{C}{\leq}$ coincide on $\mathbf{T}_n^2$ we will denote them simply by $\leq$ and the cover in $\leq$ simply by $\mathcal{D}(T)$.

Let us first give the recursive description of $S\ :\ S > T$ and of $\mathcal{D}(T)$ for $T \in \mathbf{T}_n^2$.

**Proposition.** *Let $T \in \mathbf{T}_n^2$. One has*

*(i) If $c_T(n) = 1$ then $S > T$ iff $S = n \Rightarrow S'$ where $S' > \pi_{1,n-1}(T)$. In particular, $\mathcal{D}(T) = \{n \Rightarrow S\}_{S \in \mathcal{D}(\pi_{1,n-1}(T))}$.*

*(ii) If $c_T(n) = 2$ then $S > T$ in the next two cases. Either $S = n \Rightarrow S'$ where $S' \geq \pi_{1,n-1}(T)$ or $S = \omega_1(T) \Rightarrow S'$ where $S' > T \Leftarrow n$. In particular, $\mathcal{D}(T) = \{\omega_1(T) \Rightarrow (n \Rightarrow S'')\}_{S'' \in \mathcal{D}(T_1 \setminus \{om_1(T)\}, T_2 \setminus \{n\})} \cup \{(T_1 + \{n\}, T_2 \setminus \{n\})\}$.*

*In particular, for any $T \in \mathbf{T}_n^2$ and for any $S \in \mathcal{D}(T)$ one has $\mathrm{sh}\,(S) = (\lambda_1(T) + 1, \lambda_2(T) - 1)$.*

**Proof.**
Indeed, if $c_T(n) = 1$ and $S > T$ then by lemma 3.2 one has $c_S(n) = 1$. Thus, $w_T = [n, w_{\pi_{1,n-1}(T)}]$ and $w_S = [n, w_{\pi_{1,n-1}(S)}]$. One has by lemma 2.2 (i) that $w_T \overset{D}{<} w_S$ iff $w_{\pi_{1,n-1}(T)} \overset{D}{<} w_{\pi_{1,n-1}(S)}$ which is equivalent by theorem 3.11 and its corollaries to $\pi_{1,n-1}(T) < \pi_{1,n-1}(S)$. Now if $c_T(n) = 1$ then $\mathrm{sh}\,(\pi_{1,n-1}(T)) = (\lambda_1(T) - 1, \lambda_2(T))$ and for any $S \in \mathcal{D}(T)$ one has $\mathrm{sh}\,(\pi_{1,n-1}(S)) = (\lambda_1(S) - 1, \lambda_2(S))$. Note that $\pi_{1,n-1}(S) > \pi_{1,n-1}(T)$ by shape consideration. The same shape considerations show that if $S \in \mathcal{D}(T)$

then $\pi_{1,n-1}(S) \in \mathcal{D}(\pi_{1,n-1}(T))$ and that for any $S' \in \mathcal{D}(\pi_{1,n-1}(T))$ one has $n \Rightarrow S' \in \mathcal{D}(T)$.

Now assume that $c_T(n) = 2$. Consider $S \ : \ S > T$. If $c_S(n) = 2$ then by lemma 3.10 and corollary 3.14 $(T \Leftarrow n) < (S \Leftarrow n)$. Thus, $S = \omega_1(T) \Rightarrow S'$ where $S' > T \Leftarrow n$. If $c_S(n) = 1$ then by lemma 3.4 and corollary 3.14 $S \geq T' = (T_1 + \{n\}, T_2 \setminus \{n\})$. Thus, by (i) $S = n \Rightarrow S'$ where $S' \geq \pi_{1,n-1}(T)$. If $S \in \mathcal{D}(T)$ then

(a) If $c_S(n) = 1$ then by lemma 3.4 $S = (T_1 + \{n\}, T_2 \setminus \{n\})$

(b) If $c_S(n) = 2$ then by lemma 3.10 and (i) $S = \omega_1(T) \Rightarrow (n \Rightarrow S'')$ where $S'' \in \mathcal{D}(T_1 \setminus \{\omega_1(T)\}, T_2 \setminus \{n\})$.

The note on the shape of $S \in \mathcal{D}(T)$ is obvious.

∎

**3.16** Let us give explicit description of $\mathcal{D}(T)$. Consider $T \in \mathbf{T}_n^2$. One can write $T_2$ as the union of connected subsequences $\langle T_2 \rangle = \{a_1, a_1+1, \ldots a_1+k_1\} \cup \ldots \cup \{a_s, \ldots, a_s+k_s\}$ where $a_i > a_{i-1} + k_{i-1} + 1$ for any $i \ : \ 1 < i \leq s$. For any $x \in \langle T_2 \rangle$ set $T\langle x \rangle := (T_1 + \{x\}, T_2 \setminus \{x\})$. Note that $T\langle x \rangle$ is always a tableau. Recall notion of $T\{x\}$ from 3.5. Note that for $x \in \langle T_2 \rangle$ sometimes $T\{x\} = \pi_{1,x}(T)$ and sometimes $T\{x\} \neq \pi_{1,x}(T)$. Returning to example 3.5, we get $T\{6\} = \pi_{1,6}(T)$ and $T\{5\} \neq \pi_{1,5}(T)$, $T\{3\} \neq \pi_{1,3}(T)$.

**Proposition.** *For $T \in \mathbf{T}_n^2$ let $T_2$ be the union of connected subsequences $\{a_1, a_1 + 1, \ldots a_1 + k_1\}, \ldots, \{a_s, \ldots, a_s + k_s\}$ where $a_i > a_{i-1} + k_{i-1} + 1$ for any $i \ : \ 1 < i \leq s$. Then*

$$\mathcal{D}(T) = \{T\langle a_j + k_j \rangle \mid 1 \leq j \leq s \text{ and } \pi_{1,a_j+k_j}(T) = T\{a_j + k_j\}\}.$$

**Proof.**

By corollary 3.3 (ii) and proposition 3.15, one has $\mathcal{D}(T) \subset \{T\langle s \rangle\}_{s \in T_2}$. Moreover, for any $s \ : a_j \leq s < a_j + k_j$ one has $s \in \tau(T)$ and $s \notin \tau(T\{s\})$. Thus, $T\langle s \rangle \not> T$. We obtain that $\mathcal{D}(T) \subset \{T\langle a_j + k_j \rangle\}_{j=1}^{s}$.

Consider $T' = T\langle a_j + k_j \rangle$. Since $T'_1 = T_1 + \{a_j + k_j\}$ and respectively $\langle T'_2 \rangle \subset \langle T_2 \rangle$ it is enough to show that the second condition of lemma 3.9 is satisfied, i.e. for any $s \in T'_2$ one has $s^{T'\{s\}} \neq a_j + k_j$. Indeed, if $\pi_{1,a_j+k_j}(T) = T\{a_j + k_j\}$ one has that $s^{T'\{s\}} > a_j + k_j$ for any $s > a_j + k_j$ and $s^{T'\{s\}} < a_j + k_j$ for any $s < a_j + k_j$.

On the other hand, if $T\{a_j + k_j\} \neq \pi_{1,a_j+k_j}(T)$ that means that for $a_{j+1}$ one has $a_{j+1}^{T\{a_{j+1}\}} < a_j + k_j$ Thus, $a_{j+1}^{T'\{a_{j+1}\}} = a_j + k_j$. Hence, by lemma 3.9 $T' \not> T$. And this concludes the proof.

∎

Again consider $T$ from example 3.5. One has

$$\mathcal{D}\left(\begin{array}{|cc|} 1 & 3 \\ 2 & 5 \\ 4 & 6 \\ 7 & \end{array}\right) = \left\{\begin{array}{|cc|} 1 & 3 \\ 2 & 5 \\ 4 & \\ 6 & \\ 7 & \end{array}\right\}$$

**3.17** Finally, let us consider the case of tableaux with two rows. Let $(\mathbf{T}_n^2)^\dagger$ denote the set of standard Young tableaux with two rows. For any $S \in (\mathbf{T}_n^2)^\dagger$ one has by 2.14 $S = T^\dagger(w_{S^\dagger}) = T(\overline{w}_{S^\dagger})$.

By 1.7, for any $T, S \in (\mathbf{T}_n^2)^\dagger$ on has $T \overset{D}{<} S$ (resp. $T \overset{C}{<} S$) iff $T^\dagger \overset{D}{>} S^\dagger$ (resp. $T^\dagger \overset{C}{>} S^\dagger$).

For any $S \in (\mathbf{T}_n^2)^\dagger$ set $w_S := \overline{w}_{S^\dagger}$. By 2.2 (iii), for any $S, T \in (\mathbf{T}_n^2)^\dagger$ one has $w_S \overset{D}{<} w_T$ iff $w_{S^\dagger} \overset{D}{>} w_{T^\dagger}$, therefore, all the results for $\mathbf{T}_n^2$ can be translated to $(\mathbf{T}_n^2)^\dagger$.

**Theorem.** *Let $T, S \in (\mathbf{T}_n^2)^\dagger$.*

*(i) One has $T \overset{C}{<} S$ iff $w_T \overset{D}{<} w_S$.*

*(ii) One has $T \overset{D}{<} S$ iff $w_T \overset{D}{<} w_S$.*

*(iii) $\mathcal{V}_S \subset \overline{\mathcal{V}}_T$ iff $\mathfrak{n} \cap^{w_S} \mathfrak{n} \subset \mathfrak{n} \cap^{w_T} \mathfrak{n}$.*

*(iv) Orders $\overset{D}{\leq}$ and $\overset{C}{\leq}$ coincide on $(\mathbf{T}_n^2)^\dagger$.*

**Acknowledgments.** The problem of combinatorial description of inclusion of orbital variety closures in terms of Young tableaux as well as the idea of the chain order and induced right weak order on Young tableaux were suggested by A. Joseph. I would like to thank him for this and for the fruitful discussions through various stages of this work.

I would also like to express my gratitude to the referee. His numerous remarks helped to improve the notation, alter some proofs and bring this paper to a more orthodox, and, hopefully, readable form.

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